Краткие сообщения ОИЯИ № 2(22)-87 *JINR Rapid Communications No. 2(22)-87*
УДК 517.95.951, 539.19.192

REGULARIZATION OF INTEGRODIFFERENTIAL FADDEEV EQUATIONS OVER ANGULAR VARIABLES

V.V.Pupyshev

A system of integrodifferential Faddeev equations is explored. A spectral representation of its nonlocal operators and an angular asymptotics of partial components of the wave function are found. A regularization of the initial equations at singular points of the angular part of a free three-particle Hamiltonian is performed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Регуляризация интегродифференциальных уравнений Фаддеева по угловым переменным

В.В.Пупышев

Исследуется система интегродифференциальных уравнений Фаддеева. Найдено спектральное представление ее интегральных операторов и асимптотик парциальных компонент волновой функции по угловой переменной. Осуществлена регуляризация исходных уравнений в особых точках угловой части свободного трехчастичного гамильтониана.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

1. INTRODUCTION

Differential equations [1]

$$(H_0 - E + V_i)\Psi_i = -V_i(\Psi_j + \Psi_k), \quad (i\,j\,k) = (123), (231), (312) \qquad (1)$$

for Faddeev components Ψ_i of the wave function $\Psi = \Psi_i + \Psi_j + \Psi_k$ are intensively utilized now for studying different properties of three-particle nuclear systems. Equations (1), by decomposing the wave-function-components

$$\Psi_i(\vec{x}_i, \vec{y}_i) = \sum_{aL} \Phi_i^{aL}(x_i, y_i)(x_i y_i)^{-1} \mathcal{Y}_a^{LM}(\hat{y}_i, \hat{x}_i), \; i = 1, 2, 3 \qquad (2)$$

over the bispherical basis [2]

$$\mathcal{Y}_{\alpha}^{LM}(\hat{y},\hat{x}) = \sum_{m} C_{\lambda m \ell n}^{LM} Y_{\lambda m}(\hat{y}) Y_{\ell n}(\hat{x})$$

are reduced [3] to a system of integrodifferential equations. Its solution, i.e. partial components $\Phi_i^{\alpha L}(x_i, y_i)$, are functions only of two variables, in contrast to components $\Psi_i(\vec{x}_i, \vec{y}_i)$ depending on six variables. In the polar coordinates $\rho = (x_i^2 + y_i^2)^{1/2}$, $\phi_i = \mathrm{arctg}(y_i/x_i)$ the system of integrodifferential equations has the following form

$$[\partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\hat{\Delta}_{\phi_i}^{\alpha} + E - V_i(\rho\cos\phi_i)]\Phi_i^{\alpha L}(\rho,\phi_i) =$$

$$= V_i(\rho\cos\phi_i)\sum_{k\neq i}\sum_{\alpha'} \langle \phi_i | \hat{h}_{\alpha\alpha'}^{L} | \Phi_k^{\alpha' L}(\rho,\phi_k)\rangle, \qquad (3)$$

$$i = 1,2,3,\quad \alpha = (\lambda,\ell),\quad \alpha' = (\lambda',\ell'),\quad \vec{\lambda} + \vec{\ell} = \vec{L} = \vec{\lambda}' + \vec{\ell}',$$

$$(\rho,\phi_i) \in R_+^2 = \{(\rho,\phi) : 0 \le \rho \le \infty,\ 0 \le \phi \le \pi/2\}.$$

Here we assume that the two-body potentials are central, i.e. $V_i(\vec{x}_i) = V_i(x_i)$ and take for the angular part of the two-dimensional Laplacian the following notation

$$\hat{\Delta}_{\phi}^{\alpha} \equiv \partial_\phi^2 - \lambda(\lambda+1)/\sin^2\phi - \ell(\ell+1)/\cos^2\phi . \qquad (4)$$

The requirement [3] $\Psi_i \in C_{R^6}^2$ and representation (2) provide the boundary conditions

$$\Phi_i^{\alpha L}(\rho,\phi_i) = 0,\quad \rho \in [0,\infty],\ \phi_i = 0,\pi/2;\ \rho = 0,\ \phi_i \in [0,\pi/2]. \qquad (5)$$

The boundary conditions at points $(\rho = \infty, \phi_i)$ are determined by the total energy E of a three-particle system under consideration [4]. In the framework of equations (3) a numerical investigation of three-charge particle systems (for example $dd\mu$, $dT\mu$, ...) is a very complicated problem. One difficulty is due to the Coulomb potentials. They act in all the waves, and therefore the number of partial components $\Phi_i^{\alpha L}$ taken into account, or the rank of the truncated system (3), must be sufficiently large. Another is the necessity of knowing an exact form at the asymptotic behavior of partial components near all the boundaries of the R_+^2 region. The latter is important to guarantee the required accuracy of calculations of properties of three-charge particle systems.

For this reason, a regularization of equations (3) is an urgent problem. The first step along this line is made in the present work. It consists in finding leading asymptotic terms of the functions $\Phi_i^{\alpha L}$ on the axis $\phi = 0, \pi/2$ and in regularizing equations (3) at singular points $\phi = 0, \pi/2$ of operators (4).

2. HYPERHARMONIC APPROACH AND INTEGRODIFFERENTIAL EQUATIONS

If $\Phi_i^{\alpha L} \in C^2_{R^2_+}$ and the two-body potentials are nonsingular, i.e. $V_i \in C^2_{(0,\infty)}$, $\lim_{x_i \to 0} x_i^2 V_i(x_i) = 0$, then the leading asymptotic terms of partial components at the boundaries are determined by singularities of operators (4). Therefore, the following representations

$$\Phi_i^{\alpha L}(\rho, \phi_i) = (\sin\phi_i)^{\lambda+1} (\cos\phi_i)^{\ell+1} S_i^{\alpha L}(\rho, \phi_i),\ i = 1,2,3, \tag{6}$$

take place, and regular solutions [5]

$$W_{\alpha K}(\phi) = N_{\alpha K} (\sin\phi)^{\lambda+1} (\cos\phi)^{\ell+1} P_n^{(\lambda+1/2, \ell+1/2)} (\cos 2\phi), \tag{7}$$

$n = 0, 1, \dots,\ K = 2n + \lambda + \ell$

of the boundary value problem

$$\begin{aligned} &[\hat\Delta_\phi^\alpha + (K+2)^2] W_{\alpha K}(\phi) = 0, \quad \phi \in [0, \pi/2], \\ &W_{\alpha K}(\phi) = 0, \quad \phi = 0, \pi/2, \end{aligned} \tag{8}$$

compose an orthogonal angular basis of the problem (3), (5). Equations (3), by decomposing the partial components

$$\Phi_i^{\alpha L}(\rho, \phi_i) = \sum_{n=0}^{\infty} f_{in}^{\alpha L}(\rho) W_{\alpha K}(\phi_i),\ i = 1,2,3,\ K = 2n + \lambda + \ell, \tag{9}$$

over basis (1), reduce to the system of ordinary second-order differential equations

$$\sum_{m=0}^{\infty} \{[\partial_\rho^2 + \rho^{-1}\partial_\rho + E - \rho^{-2}(K+2)^2]\delta_{nm} - V_{inm}^{\alpha}(\rho)\} f_{im}^{\alpha L}(\rho) =$$

$$= \sum_{m,p=0}^{\infty} V^{a}_{inm}(\rho) \sum_{k \neq i} \sum_{a'} \langle W_{aK'} | \hat{h}^{L}_{aa'} | W_{a'K''} \rangle f^{a'L}_{kp}(\rho),$$

$$K = 2n + \lambda + \ell, \; K' = 2m + \lambda + \ell', \; K'' = 2p + \lambda' + \ell', \tag{10}$$

$$V^{a}_{inm} = \langle W_{aK} | V_i | W_{aK'} \rangle,$$

for unknown radial functions $f^{aL}_{in}(\rho)$, $n = 0, 1, \ldots$

Notice that the sequence of series (2), (9) is equivalent to the expansion of components

$$\Psi_i(\vec{x}_i, \vec{y}_i) = \sum_{aLK} f^{aL}_{in}(\rho)\rho^{-2} U^{aL}_{K}(\phi_i, \hat{y}_i, \hat{x}_i), \; i = 1, 2, 3, \tag{11}$$

over polyspherical hyperharmonics

$$U^{aL}_{K}(\phi, \hat{y}, \hat{x}) = 2 \operatorname{cosec} 2\phi W_{aK}(\phi) \mathcal{Y}^{LM}_{a}(\hat{y}, \hat{x}).$$

Let us apply the hyperharmonic approach not to the Schrödinger three-particle equation, as it is usually done, but to its Faddeev splitting (1), i.e. substitute the components (11) into the system (1). As a result, we obtain a set of equations whose left-hand side is identical with the one of set (10) and the right part contains, instead of the matrix elements of $\hat{h}^{L}_{aa'}$-operators, the well-known Rejnal-Revai [6] coefficients $\langle a | a' \rangle_{KL}$. The equalities

$$\langle W_{aK} | \hat{h}^{L}_{aa'} | W_{a'K'} \rangle = \delta_{KK'} \langle a | a' \rangle_{KL},$$

thus proved and the completeness of the function set (7) provide us with the spectral representations of the nonlocal operators

$$\hat{h}^{L}_{aa'} = \sum_{K} | W_{aK} \rangle \langle a | a' \rangle_{KL} \langle W_{a'K} | . \tag{12}$$

Formulae (10)-(12) generalise the results of works [7] and the previous author's work [8] to the case of arbitrary indices a, a', L and particle masses.

3. REGULAR OVER ANGULAR VARIABLES

Now we substitute the partial components (6) into the system (3) and introduce new angular variables $v_i = \cos 2\phi_i$. As a result, we obtain the set of equations

$$\{\partial_\rho^2 + \rho^{-1}\partial_\rho + \rho^{-2}\,[\,4(1 - v_i^2\,)\partial^2_{v_i} - 4(\lambda - \ell + (3 + \lambda + \ell)v_i)\partial_{v_i} -$$

$$- (\lambda + \ell + 2\,)^2] + E - V_i\,(\rho, v_i\,)\}\,S_i^{\alpha L}\,(\rho\,, v_i\,) = \qquad (13)$$

$$= V_i\,(\rho, v_i)\sum_{k\neq i}\sum_{\alpha'} \langle v_i\,|\hat{\tilde{h}}^{L}_{\alpha\alpha'}|\,S_k^{\alpha' L}\,(\rho, v_k\,)\rangle, \quad i = 1, 2, 3.$$

for new unknown functions $S_i^{\alpha L}\,(\rho, \pm 1) \neq 0$. They have no zeros of order λ and ℓ of the boundaries $v_i = +1, -1$, respectively, in contrast to the partial components $\Phi_i^{\alpha L}$. The kernels of the new operators

$$\hat{\tilde{h}}^{L}_{\alpha\alpha'} = \tfrac{1}{4}\,[\,(\sin\phi_i\,)^{\lambda+1}\,(\cos\phi_i\,)^{\ell+1}\,]^{-1}\,\times$$

$$\times\; h^{L}_{\alpha\alpha'}[\,(\sin\phi_k)^{\lambda'}\,(\cos\phi_k\,)^{\ell'}\,],$$

owing to equalities (7) and (12) are regular functions. The angular part of the new Laplacian does not contain singular operators, in contrast to the old (4). Thus, we perform a regularization (13) over angular variables of equations (3).

A next, more complicated step is a regularization of equation (3) over the radial variable. Investigation of the set (10) in a vicinity of its singular point $\rho = 0$ is one possible way for solving this important problem.

4. CONCLUSION

In conclusion we briefly summarize main results of the present work. The spectral representation of the nonlocal operators $\hat{h}^{L}_{\alpha\alpha'}$ is found for arbitrary indices α, α' and L. The connection (9-12) between the integrodifferential three-particle approach and the hyperharmonic one is established in a most general case. Equations (13) regular over angular variables are constructed.

REFERENCES

1. Noyes H.P. — In: "Three-Body Problem in Nuclear and Particle Physics", North Holland, Amsterdam, 1970, p.2.

2. Варшалович Д.А., Москалев Н.А., Херсонский В.К. — "Квантовая теория углового момента", "Наука", Л., 1975.
3. Merkyriev S.P., Gignoux C., Laverne A. — Ann.Phys., 1976, 99, p.30.
4. Меркурьев С.П., Фаддеев Л.Д. — "Квантовая теория рассеяния для систем нескольких частиц", "Наука", М., 1985.
5. Абрамовиц М., Стиган И. — "Справочник по специальным функциям", "Наука", М., 1979.
6. Rejnal J., Revai J. — Nuovo Cimento, 1970, 68A, p.612.
7. Mandelzveig V.B. — Ann.Phys., 1977, 104, p.1.
M.Fabre de la Ripelle. — Few-Body System, 1986, 1, p.181.
8. Пупышев В.В — ЯФ, 1986, 43, с.420, Препринт ОИЯИ, Е4-84-808, Дубна, 1984.

Received on February 24, 1987.